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ON THE STABILITY OF INHOMOGENEOUSLY AGEING VISCOELASTIC PLATES*

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A method of investigation is proposed and conditions are set up for the stability of viscoelastic inhomogeneously ageing plates of arbitrary shape with a common creep kernel. The form of the stability conditions is found as a function of the surface forces. The stability problem is examined numerically in a finite time interval. The paper touches on the investigations in /1—3/. (See the bibliography of research on the stability of homogeneous viscoelastic systems in /1—5/, for example.)

1. Formulation of the problem. We consider a viscoelastic inhomogeneously ageing plate of constant thickness h. We introduce a Cartesian $Ox_1x_2x_3$ coordinate system whose Ox_3 axis is perpendicular to the middle plane while the Ox_1x_2 plane agrees with the middle plane in the undeformed state. The plate is a set of points $\{x=(x_1, x_2)\in D\times(-h/2\leqslant x_3\leqslant h/2)\}$, where D is the domain in the Ox_1x_2 plane. The domain D has a piecewise-smooth boundary $\Gamma=\partial D$.

At the time $t=0$ a stationary external load consisting of a distributed transverse load of intensity $-q(x)$ applied to the face $x_3=h/2$, and the surface forces $F(x)=(F_1(x), F_2(x), 0)$, applied to the part of the plate edge $\Gamma_1\times[-h/2, h/2], \Gamma_1\subset\Gamma$, is applied to the undeformed plate. The growth of a plate element x with respect to the plate element $x=0$ is $\rho(x)$, where the function $\rho(x)$ is piecewise continuous and bounded. The displacement vector in the plane of the plate is denoted by (u_1^0, u_2^0), and the displacement (deflection) of points of the middle plane in the direction of the Ox_3 axis is denoted by u. The values of the vector (u_1^0, u_2^0) for points of the middle plane are denoted by (u_1, u_2). There are no displacements in the plane of the plate on part of the plate edges $\Gamma_2\times[-h/2, h/2]$, $\Gamma_2=\Gamma\setminus\Gamma_1$

$$u_i^0(x)=0,\ x\in\Gamma_2,\ i=1,\ 2 \tag{1.1}$$

The plate edge is clamped with respect to deflection, i.e., when the following Kirchhoff hypotheses are satisfied:

$$u(x)=0,\ u_{,n}(x)=0,\ x\in\Gamma \tag{1.2}$$

Here n is the unit vector of the external normal to the curve Γ in the plane of the plate and denoted by $f_{,i}=\partial f/\partial x_i$.

Definition. The plate is called stable if for any $\varepsilon>0$ there is a $\delta=\delta(\varepsilon)>0$ such that from the inequality $|q(x)|<\delta$ there results the estimate $|u(t, x)|<\varepsilon$, $t\geqslant 0$, $x\in D$.

2. Governing equations. Let $\sigma^\circ=(\sigma_{ij}^\circ)$, $\varepsilon^\circ=(\varepsilon_{ij}^\circ)$, $i, j=1, 2, 3$ denote the stress and strain tensors at the time $t\geqslant 0$. According to the model of inhomogeneously ageing bodies, the tensor components σ° and ε° are connected by the relationships

$$\varepsilon_{ij}^\circ=\frac{1}{E}(I+K)[(1+\nu)\sigma_{ij}^\circ-\nu\delta_{ij}\sigma_{ll}^\circ] \tag{2.1}$$
$$\sigma_{ij}^\circ=\frac{E}{1+\nu}(I-R)\left[\varepsilon_{ij}^\circ-\frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{ll}^\circ\right]$$

Here E is the constant elastically-instantaneous strain modulus, ν is the constant Poisson's ratio, summation is over identical subscripts, δ_{ij} is the Kronecker delta, I is the unit operator, K is the creep operator, R is the relaxation operator ($k(t, \tau)$ is the creep kernel, and $r(t, \tau)$ is the relaxation kernel)

$$K\sigma_{ij}^\circ=\int_0^t k(t+\rho(x),\quad \tau+\rho(x))\,\sigma_{ij}^\circ\,d\tau$$
$$R\varepsilon_{ij}^\circ=\int_0^t r(t+\rho(x),\quad \tau+\rho(x))\,\varepsilon_{ij}^\circ\,d\tau$$
$$I-R=(I+K)^{-1}$$

It is assumed that the functions $k(t, \tau)$, $r(t, \tau)$ are weakly singular, the Kirchhoff hypotheses are satisfied /6/, the elongations and shears are small compared with unity, and the squares of the rotations are small compared with the elongations and shears. Then the

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strain tensor components ε_{ij}°, $i, j = 1, 2$, have the form

$$\varepsilon_{ij}^{\circ} = \varepsilon_{ij} - x_3 u_{,ij} \tag{2.2}$$

Here ε_{ij} are the strain tensor components in the middle plane. Moreover, it is assumed that a function $r_1(t, \tau)$ exists such that for any $x \in D$

$$0 \leqslant r(t + \rho(x),\ \tau + \rho(x)) \leqslant r_1(t,\ \tau),\ 0 \leqslant \tau \leqslant t \tag{2.3}$$

$$|r_1| = \sup_t \int_0^t r_1(t, \tau)\, d\tau < 1 \tag{2.4}$$

The stresses σ_{i3}°, $i = 1, 2, 3$ can be neglected in (2.1) in the generalized plane state of stress. Hence, and from (2.1) it follows that

$$\begin{aligned} \sigma_{11}^{\circ} &= E(1 - \nu^2)^{-1}(I - R)(\varepsilon_{11}^{\circ} + \nu\varepsilon_{22}^{\circ}), \\ \sigma_{12}^{\circ} &= E(1 + \nu)^{-1}(I - R)\,\varepsilon_{12}^{\circ} \\ \tau_{22}^{\circ} &= E(1 - \nu^2)^{-1}(I - R)(\nu\varepsilon_{11}^{\circ} + \varepsilon_{22}^{\circ}) \end{aligned} \tag{2.5}$$

Let σ_{ij} and M_{ij} denote the average forces and moments in the plate per unit area of cross-section perpendicular to the middle plane

$$\sigma_{ij} = \frac{1}{h}\int_{-h/2}^{h/2} \sigma_{ij}^{\circ}\, dx_3,\quad M_{ij} = \frac{1}{h}\int_{-h/2}^{h/2} \sigma_{ij}^{\circ} x_3\, dx_3 \tag{2.6}$$

It is clear that $\varepsilon_{ij}(t, x)$ and $u_i(t, x)$ are the values of the strain ε_{ij}° and the displacement u_i° averaged over the plate thickness. The averaged quantities are functions of the variables t and x, on which the explicit dependence is sometimes either not indicated or else indicated only for one of them.

Substituting (2.2), (2.5) for ε_{ij}°, σ_{ij}° into the relationships (2.6), we obtain

$$\begin{aligned} \sigma_{11} &= E(1 - \nu^2)^{-1}(I - R)(\varepsilon_{11} + \nu\varepsilon_{22}) \\ \sigma_{12} &= E(1 + \nu)^{-1}(I - R)\,\varepsilon_{12} \\ \sigma_{22} &= E(1 - \nu^2)^{-1}(I - R)(\nu\varepsilon_{11} + \varepsilon_{22}) \end{aligned} \tag{2.7}$$

$$\begin{aligned} M_{11} &= -Eh^2[12(1 - \nu^2)]^{-1}(I - R)(u_{,11} + \nu u_{,22}) \\ M_{12} &= -Eh^2[12(1 + \nu)]^{-1}(I - R)\,u_{,12} \\ M_{22} &= -Eh^2[12(1 - \nu^2)]^{-1}(I - R)(\nu u_{,11} + u_{,22}) \end{aligned} \tag{2.8}$$

Usting /7/, we write the plate equilibrium equations in the bent state in the form

$$\sigma_{11,1} + \sigma_{12,2} = 0,\ \sigma_{12,1} + \sigma_{22,2} = 0 \tag{2.9}$$

$$\begin{aligned} &M_{11,1} + M_{12,2} - \sigma_{13} = 0,\ M_{12,1} + M_{22,2} - \sigma_{23} = 0 \\ &\sigma_{13,1} + \sigma_{23,2} + q/h + (\sigma_{11}u_{,1})_{,1} + (\sigma_{12}u_{,2})_{,1} + (\sigma_{12}u_{,1})_{,2} + (\sigma_{22}u_{,2})_{,2} = 0 \end{aligned} \tag{2.10}$$

To eliminate the forces σ_{13}, σ_{23} from (2.10), we differentiate the first equation in (2.10) with respect to x_1, the second with respect to x_2, and add to the third equation. Substituting their expressions (2.8) in place of the moments M_{ij} in the result, we arrive at the governing equation for the displacement $u(t, x)$

$$\begin{aligned} &[(I - R)(u_{,11} + \nu u_{,22})]_{,11} + 2(1 - \nu)[(I - R)u_{,12}]_{,12} + \\ &[(I - R)(\nu u_{,11} + u_{,22})]_{,22} = q/\beta + h/\beta\,[(\sigma_{11}u_{,1})_{,1} + \\ &(\sigma_{12}u_{,2})_{,1} + (\sigma_{12}u_{,1})_{,2} + (\sigma_{22}u_{,2})_{,2}],\quad \beta = Eh^3/[12(1 - \nu^2)] \end{aligned} \tag{2.11}$$

Here β is the cylindrical stiffness of the plate and the stresses σ_{ij} are determined by (2.7).

3. General stability conditions.

We will derive the stability conditions by considering the functions σ_{ij} as given. We will estimate the displacement u as a function of the force σ_{ij} by using the notation $u_{,l}^2 = (u_{,l})^2$.

It follows from (1.2) that

$$u = 0,\ u_{,1} = 0,\ u_{,2} = 0,\ x \in \Gamma \tag{3.1}$$

We multiply (2.11) by $u(t, x)$ and integrate over the domain D. Taking account of the Green's formula (/8/, p.69) and the boundary conditions (3.1), we obtain

$$\begin{aligned} &\int_D [u_{,11}(I - R)(u_{,11} + \nu u_{,22}) + 2(1 - \nu)u_{,12}(I - R)u_{,12} + \\ &u_{,22}(I - R)(\nu u_{,11} + u_{,22})]\, dx = \\ &\frac{1}{\beta}\int_D qu\, dx - \frac{h}{\beta}\int_D (\sigma_{11}u_{,1}^2 + 2\sigma_{12}u_{,1}u_{,2} + \sigma_{22}u_{,2}^2)\, dx \end{aligned} \tag{3.2}$$

We introduce the functions

$$f_0^2(t)=\int_D (u_{,1}^2+u_{,2}^2)^2\,dx,\quad f_1^2(t)=\int_D u^2\,dx$$

$$H^2(t)=\int_D (\sigma_{11}^2+2\sigma_{12}^2+\sigma_{22}^2)\,dx$$

Using Green's formula and the boundary conditions (3.1), we conclude that for any ν

$$f^2(t)=\int_D (u_{,11}^2+2u_{,12}^2+u_{,22}^2)\,dx=\int_D (u_{,11}+u_{,22})^2\,dx=$$
$$(1-\nu)\int_D (u_{,11}^2+2u_{,12}^2+u_{,22}^2)\,dx+\nu\int_D (u_{,11}+u_{,22})^2\,dx$$

We now represent (3.2) in the form

$$f^2(t)=\frac{1}{\beta}\int_D qu\,dx+(1-\nu)\int_D (u_{,11}Ru_{,11}+2u_{,12}Ru_{,12}+ \tag{3.3}$$
$$u_{,22}Ru_{,22})\,dx+\nu\int_D (u_{,11}+u_{,22})\,R\,(u_{,11}+u_{,22})\,dx-$$
$$\frac{h}{\beta}\int_D (\sigma_{11}u_{,1}^2+2\sigma_{12}u_{,1}u_{,2}+\sigma_{22}u_{,2}^2)\,dx=I_1+(1-\nu)I_2+\nu I_3+I_4$$

It is clear that

$$|I_1|\leqslant\|q\|f_1(t),\quad \|q\|^2=\frac{1}{\beta^2}\int_D q^2(x)\,dx$$

Furthermore, using the Cauchy inequality, we have

$$|I_2|\leqslant\int_0^t r_1(t,\tau)\,d\tau\int_D (|u_{,11}(t,x)\,u_{,11}(\tau,x)|+2\,|u_{,12}(t,x)\times \tag{3.4}$$
$$u_{,12}(\tau,x)|+|u_{,22}(t,x)\,u_{,22}(\tau,x)|)\,dx\leqslant f(t)\int_0^t r_1(t,\tau)\,f(\tau)\,d\tau$$

Similarly

$$|I_3|\leqslant f(t)\int_0^t r_1(t,\tau)\,f(\tau)\,d\tau \tag{3.5}$$

Like (3.4) we have

$$|I_4|\leqslant\frac{h}{\beta}\int_D (\sigma_{11}^2+2\sigma_{12}^2+\sigma_{22}^2)^{1/2}(u_{,1}^2+u_{,2}^2)\,dx\leqslant\frac{h}{\beta}H(t)\,f_0(t)$$

Hence, and from (3.3) it follows that

$$f^2(t)\leqslant\|q\|f_1(t)+f(t)\int_0^t r_1(t,\tau)\,f(\tau)\,d\tau+\frac{h}{\beta}H(t)\,f_0(t) \tag{3.6}$$

Let λ_1 and λ denote the greatest positive numbers for which the inequalities

$$f_1^2(t)\leqslant\lambda_1^{-2}f^2(t),\quad f_0(t)\leqslant\lambda^{-1}f^2(t) \tag{3.7}$$

hold for all functions $u(x)\not\equiv 0$ satisfying the boundary conditions (3.1).
The existence of such λ_1 and λ is proved in Sect.5.
Because of (3.7), the following inequality results from estimate (3.6):

$$(1-h\,\|H\|\,\lambda^{-1}\beta^{-1})\,\|f(t)\|\leqslant|r_1|\,\|f(t)\|+\lambda_1^{-1}\,\|q\| \tag{3.8}$$
$$\|H\|=\sup^{t\geqslant 0}H(t),\quad \|f(t)\|=\sup_{0\leqslant\tau\leqslant t}f(\tau)$$

We now assume that

$$\|H\|<\beta\lambda h^{-1}(1-|r_1|) \tag{3.9}$$

On the basis of (3.8) and (3.9) we have $\|f(t)\|\leqslant c_1\|q\|$. Here and henceforth $c_i>0$ are certain constants. Because of (3.7) and the known results (/9/, p.84, inequality (8.4)), we have $|u(t,x)|\leqslant c_2\|f(t)\|$.
Therefore, we have

Theorem 3.1. Let the assumptions formulated in Sect.1 and the estimates (2.3), (2.4), (3.9) be satisfied. Then the plate is stable.
Using the method to prove Theorem 3.1, other stability conditions can be obtained as in /3/; for instance, the following theorem holds:

Theorem 3.2. We assume that the relaxation kernel r satisfies inequality (2.3), where $|r_1|<\infty$. Furthermore, let there be a function $r_0(t,\tau)$, $|r_0|<1$ such that uniformly in $t\geqslant T$

$$\lim_{T\to\infty}\int_T^t \sup_{x\in D}|r(t+\rho(x),\tau+\rho(x))-r_0(t,\tau)|\,d\tau = 0$$

Then the plate is stable if $\|H\| < \beta\lambda h^{-1}(1-|r_0|)$.

4. Specific stability conditions.

The general stability conditions set up in Theorem 3.1 and 3.2 depend on the stresses in the plate by means of the function H. In addition, stability conditions formulated directly in terms of the surface forces F are of interest.

If $\rho(x)\equiv 0$ and $\Gamma_1=\Gamma$, then the forces for a viscoelastic plate agree with the forces for a corresponding elastic plate (i.e., for $R\equiv 0$). We estimate H in terms of F for an arbitrary function ρ. To do this, $H(t)$ is estimated first in terms of the function $J(t)$ equal to

$$J^2(t)=\int_D(\varepsilon_{11}^2+2\varepsilon_{12}^2+\varepsilon_{22}^2)\,dx$$

Then $J(t)$ is estimated in terms of F. We formulate the appropriate theorem.

Theorem 4.1. If the assumptions of Theorem 3.1 are satisfied, the plate is stable for $\beta_1<\beta\lambda(1-|r_1|)h^{-1}$ where

$$\beta_1=(1+\nu)(1+|r_1|)[(1-\nu)\lambda_2(1-|r_1|)]^{-1}|F|.$$

If the assumptions of Theorem 3.2 are satisfied, then the plate is stable for $\beta_1<\beta\lambda(1-|r_0|)h^{-1}$.

Proof. To obtain the first estimate we substitute (2.7) for the stress tensor component into $H(t)$. We obtain

$$H^2 t=\frac{E}{(1-\nu^2)}\Big[(1-\nu)\int_D(\sigma_{11}(I-R)\varepsilon_{11}+2\sigma_{12}(I-R)\varepsilon_{12}+ \sigma_{22}(I-R)\varepsilon_{22})\,dx+\nu\int_D(\sigma_{11}+\sigma_{22})(I-R)(\varepsilon_{11}+\varepsilon_{22})\,dx\Big] \qquad (4.1)$$

Similarly we have

$$H_1^2(t)=\int_D(\sigma_{11}+\sigma_{22})^2dx=\int_D(\sigma_{11}+\sigma_{22})\Big[\frac{E}{1-\nu^2}(I-R)(\varepsilon_{11}+\nu\varepsilon_{22})+ \frac{E}{1-\nu^2}(I-R)(\varepsilon_{22}+\nu\varepsilon_{11})\Big]dx=\frac{E}{(1-\nu)}\int_D(\sigma_{11}+\sigma_{22})(I-R)(\varepsilon_{11}+\varepsilon_{22})\,dx$$

Hence and from (4.1) it follows that

$$H^2(t)-\frac{\nu}{1+\nu}H_1^2(t)=\frac{E}{1+\nu}Q \qquad (4.2)$$

$$Q=\int_D[\sigma_{11}(I-R)\varepsilon_{11}+2\sigma_{12}(I-R)\varepsilon_{12}+\sigma_{22}(I-R)\varepsilon_{22}]\,dx$$

The left side of (4.2) has the following expression as lower bound:

$$H^2-\frac{\nu}{1+\nu}H_1^2\geqslant\frac{1-\nu}{1+\nu}H^2 \qquad (4.3)$$

The quantity Q in (4.2) is estimated as follows:

$$|Q|\leqslant\Big|\int_D(\sigma_{11}\varepsilon_{11}+2\sigma_{12}\varepsilon_{12}+\sigma_{22}\varepsilon_{22})\,dx\Big|+ \Big|\int_0^t d\tau\int_D r(t+\rho(x),\tau+\rho(x))[\sigma_{11}(t)\varepsilon_{11}(\tau)+2\sigma_{12}(t)\varepsilon_{12}(\tau)+ \sigma_{22}(t)\varepsilon_{22}(\tau)]\,dx\Big|\leqslant H(t)J(t)+H(t)\int_0^t r_1(t,\tau)J(\tau)\,d\tau\leqslant(1+|r_1|)H(t)\|J(t)\|, \qquad (4.4)$$

$$\|J(t)\|=\sup_{0\leqslant\tau\leqslant t}J(\tau)$$

The necessary estimate results from (4.2)-(4.4)

$$(1-\nu)H(t)\leqslant E(1+|r_1|)\,\|J(t)\| \qquad (4.5)$$

We will now $\|J(t)\|$ in terms of F. In conformity with the loading conditions on Γ_1 the following boundary conditions hold:

$$\sigma_{11}\cos(n,x_1)+\sigma_{12}\cos(n,x_2)=F_1$$
$$\sigma_{12}\cos(n,x_1)+\sigma_{22}\cos(n,x_2)=F_2 \qquad (4.6)$$

We multiply the first of equations (2.9) by u_1, the second by u_2 and we add and integrate over the domain D. Taking account of the boundary conditions (1.1), (4.6) and Green's formula, we obtain

$$\int_D(\sigma_{11}\varepsilon_{11}+2\sigma_{12}\varepsilon_{12}+\sigma_{22}\varepsilon_{22})\,dx=\alpha_1,\quad \alpha_1=\int_{\Gamma_1}(F_1u_1+F_2u_2)\,ds \qquad (4.7)$$

where ds is the arc element of Γ.

Replacing the stress tensor components in (4.7) by (2.7), we obtain

$$E(1-\nu^2)^{-1}\int_D [\varepsilon_{11}(I-R)(\varepsilon_{11}+\nu\varepsilon_{22})+2(1-\nu)\varepsilon_{12}(I-R)\varepsilon_{12}+\varepsilon_{22}(I-R)(\varepsilon_{22}+\nu\varepsilon_{11})]\,dx=\alpha_1$$

It hence follows that the representation

$$J_0^2=(1-\nu)J^2+\nu J_1^2,\quad J_0^2(t)=(1-\nu^2)E^{-1}\alpha_1+(1-\nu)\alpha_2+\nu\alpha_3 \tag{4.8}$$

holds.

We here assumed

$$J_1^2(t)=\int_D(\varepsilon_{11}^2+\varepsilon_{22}^2)\,dx$$

$$\alpha_2=\int_D(\varepsilon_{11}R\varepsilon_{11}+2\varepsilon_{12}R\varepsilon_{12}+\varepsilon_{22}R\varepsilon_{22})\,dx,\qquad \alpha_3=\int_D(\varepsilon_{11}+\varepsilon_{22})R(\varepsilon_{11}+\varepsilon_{22})\,dx$$

We will estimate the individual components on the right side of (4.8). The function α_2 on the right side of (4.8) is estimated thus

$$|\alpha_2|=\Big|\int_0^t d\tau\int_D r(t+\rho(x),\tau+\rho(x))[\varepsilon_{11}(t)\varepsilon_{11}(\tau)+2\varepsilon_{12}(t)\varepsilon_{12}(\tau)+$$

$$\varepsilon_{22}(t)\varepsilon_{22}(\tau)]\,dx\Big|\leqslant\int_0^t r_1(t,\tau)\,d\tau\int_D[\varepsilon_{11}^2(t)+2\varepsilon_{12}^2(t)+\varepsilon_{22}^2(t)]^{1/2}[\varepsilon_{11}^2(\tau)+$$

$$2\varepsilon_{12}^2(\tau)+\varepsilon_{22}^2(\tau)]^{1/2}dx\leqslant J(t)\int_0^t r_1(t,\tau)J(\tau)\,d\tau$$

Analogously, for α_3 the following estimate holds:

$$|\alpha_3|\leqslant\int_0^t r_1(t,\tau)\,d\tau\int_D|\varepsilon_{11}(t)+\varepsilon_{22}(t)|\;|\varepsilon_{11}(\tau)+\varepsilon_{22}(\tau)|\,dx\leqslant J_1(t)\int_0^t r_1(t,\tau)J_1(\tau)\,d\tau$$

We now introduce the number $\lambda_2>0$ by using the equation

$$\lambda_2^2=\inf_{v_1,v_2}\int_D(e_{11}^2+2e_{12}^2+e_{22}^2)\,dx\Big/\int_\Gamma(v_1^2+v_2^2)\,ds \tag{4.9}$$

Here the functions v_1 and v_2 satisfy the boundary conditions (1.1) and we set $2e_{ij}=v_{i,j}+v_{j,i}$. From the definition of λ_2 it follows that

$$\int_{\Gamma_1}(u_1^2+u_2^2)\,ds\leqslant\lambda_2^{-2}J^2(t) \tag{4.10}$$

It hence follows that the component α_1 on the right side of (4.8) satisfies the inequality

$$|\alpha_1|\leqslant|F|\Big[\int_{\Gamma_1}(u_1^2+u_2^2)\,ds\Big]^{1/2}\leqslant|F|\,J(t)\lambda_2^{-1}$$

$$|F|^2=\int_{\Gamma_1}(F_1^2+F_2^2)\,ds$$

It follows from (4.8) and estimates set up for the quantities α_i that

$$J_0^2(t)\leqslant(1-\nu^2)[E\lambda_2]^{-1}|F|\,J(t)+\int_0^t r_1(t,\tau)[(1-\nu)J(t)J(\tau)+\nu J_1(t)J_1(\tau)]\,d\tau \tag{4.11}$$

Moreover, on the basis of the Cauchy inequality and (4.8) we have

$$(1-\nu)J(t)J(\tau)+\nu J_1(t)J_1(\tau)\leqslant J_0(t)J_0(\tau) \tag{4.12}$$

Taking account of (4.12), inequality (4.11) takes the form

$$J_0^2(t)\leqslant J_0(t)\int_0^t r_1(t,\tau)J_0(\tau)\,d\tau+(1-\nu^2)(E\lambda_2)^{-1}|F|\,J(t) \tag{4.13}$$

Furthermore, it is clear that

$$J_0(t)\int_0^t r_1(t,\tau)J_0(\tau)\,d\tau\leqslant\|J_0(t)\|^2|r_1| \tag{4.14}$$

$$\|J_0(t)\|=\sup_{0\leqslant\tau\leqslant t}J_0(\tau)$$

Hence, and from (2.2) it follows that

$$(1-|r_1|)\|J_0(t)\|^2\leqslant(1-\nu^2)(E\lambda_2)^{-1}|F|\;\|J(t)\| \tag{4.15}$$

But by virtue of the definition (4.8) of the function J_0 the inequality $\|J_0(t)\|^2\geqslant(1-\nu)\|J(t)\|^2$ holds. This means that taking (4.15) into account

$$J(t)\leqslant\|J(t)\|\leqslant(1+\nu)|F|[E\lambda_2(1-|r_1|)]^{-1} \tag{4.16}$$

The following relationship results from the estimates (4.5) and (4.16)

$$H(t) \leqslant \beta_1 = (1+\nu)(1+|r_1|)[(1-\nu)\lambda_2(1-|r_1|)]^{-1}|F|$$

Comparing this inequality with the assertions of Theorems 3.1 and 3.2, we conclude that Theorem 4.1 holds.

5. Certain remarks. 1°. We present a foundation for the positivity of the numbers $\lambda, \lambda_1, \lambda_2$ introduced above. The positivity of λ_1 follows from the fact that λ_1 is the minimum eigenvalue of the boundary value problem $\Delta\Delta u - \lambda_1 u = 0$ with boundary conditions (1.2) (Δ is the Laplace operator), because of the Rayleigh inequality (/10/, p.167).

We note for the basis of the inequality $\lambda > 0$ that (/9/, p.84)

$$\left(\int_D u_{,i}^4 dx\right)^{1/2} \leqslant c_3 \int_D (u_{,i}^2 + u_{,i1}^2 + u_{,i2}^2)\, dx \tag{5.1}$$

Moreover (/9/, p.63)

$$\int_D u_{,i}^2 dx \leqslant c_4 \int_D (u_{,i1}^2 + u_{,i2}^2)\, dx, \quad i = 1,2 \tag{5.2}$$

The constants $c_3 > 0, c_4 > 0$ in (5.1) and (5.2) depend only on the domain D. It follows from (5.1), (5.2) and the Minkowski inequality that

$$f_0(t) \leqslant \left(\int_D u_{,1}^4 dx\right)^{1/2} + \left(\int_D u_{,2}^4 dx\right)^{1/2} \leqslant c_3(1+c_4)\int_D (u_{,11}^2 + 2u_{,12}^2 + u_{,22}^2)\, dx = c_3(1+c_4) f^2(t)$$

The existence of $\lambda > 0$ is thereby established taking (3.7) into account.

We finally turn to the parameter λ_2 defined by relationship (4.9). For any $\delta > 0$ we let $D(\delta)$ denote the set of points of D that are removed by not more than δ from Γ. We have (/9/, p.73)

$$\int_\Gamma v_i^2 ds \leqslant c_5 \left[\frac{1}{\delta}\int_{D(\delta)} v_i^2 dx + \delta \int_{D(\delta)} (v_{i,1}^2 + v_{i,2}^2)\, dx\right]$$

This means that a constant c_6, which depends solely on Γ, exists such that

$$\int_\Gamma v_i v_i ds \leqslant c_6 \int_D (v_i v_i + v_{i,j} v_{i,j})\, dx, \quad i, j = 1,2 \tag{5.3}$$

Furthermore, by virtue of the Korn inequality a constant $c_7 > 0$ exists independent of v_i such that (/11, p.45)

$$\left(\int_D v_i v_i\, dx\right)^{1/2} + \left(\int_D v_{i,j} v_{i,j} dx\right)^{1/2} \leqslant c_7 \left(\int_D e_{ij} e_{ij} dx\right)^{1/2}$$

Hence, squaring both sides and comparing the result with inequality (5.4) we conclude that

$$\int_\Gamma v_i v_i ds \leqslant c_6 c_7^2 \int_D e_{ij} e_{ij} dx$$

This means that the inequality $\lambda_2 > 0$ is established because of (4.9).

2°. The stability conditions established in Theorems 3.1, 3.2, 4.1 retain their form even for other plate support methods. It is sufficient just to replace therein, the parameters λ and λ_2 which have been defined by (3.7) and (4.9), by new values corresponding to the method of support under consideration.

3°. When using the Euler approach to elastic plate stability, that plate shape and loading method are ordinarily assumed for which the stresses within the plate are constant (see /7,12/, say). Under this assumption the stability conditions are simplified and have the following form. Let $\lambda_3 - \lambda_5$ denote the minimal eigenvalues of the boundary value problems

$$\Delta\Delta u + \lambda_3 \Delta u = 0, \ \Delta\Delta u + \lambda_4 u_{,11} = 0, \ \Delta\Delta u + 2\lambda_5 u_{,12} = 0$$

with homogeneous boundary conditions corresponding to the support mode.

Theorem 5.1. Let the stresses within the plate be constant and let the assumptions of Theorem 3.1 (Theorem 3.2) be satisfied. Then if $\sigma_{11} = \sigma_{22}, \sigma_{12} = 0$ the plate is stable for $|\sigma_{11}| < z_1(\lambda_3)$ (for $|\sigma_{11}| < z_2(\lambda_3)$), if $\sigma_{12} = \sigma_{22} = 0$ the plate is stable for $|\sigma_{11}| < z_1(\lambda_4)$ (for $|\sigma_{11}| < z_2(\lambda_4)$), if $\sigma_{11} = \sigma_{22} = 0$ the plate is stable for $|\sigma_{12}| < z_1(\lambda_5)$ (for $|\sigma_{12}| < z_2(\lambda_5)$), where $z_1(\lambda) = \beta\lambda h^{-1}(1 - |r_1|)$, and $z_2(\lambda) = \beta\lambda h^{-1}(1 - |r_0|)$.

For instance, we consider an elastic rectangular plate of length a and width b compressed uniformly in all directions by a force of intensity p, hinge-supported along the outline. Then $\sigma_{11} = \sigma_{22} = -p/h, \ \sigma_{12} = 0$. The parameter $\lambda_3 = \pi^2(a^{-2} + b^{-2})$. This means that the plate is stable for $p < \pi^2\beta(a^{-2} + b^{-2})$ by virtue of Theorem 5.1, which agrees with the known result (/13/, p.462).

4°. From these theorems, certain stability conditions can be obtained as a limiting case for viscoelastic rods presented in /3/. For a rectangular plate of length a and width b, $|x_2| \leqslant b, b \gg a$, let constant compressive forces of intensity p be applied to the edges $|x_1| = a/2$.

The edges $|x_2| = b/2$ are stress free. We assume that $\nu = 0$ and $p = p(x_1)$. Then $\sigma_{11} = -p/h$, $\sigma_{12} = \sigma_{22} = 0$, and the plate deformation occurs along a cylindrical surface and is characterized by the bending of an arbitrary beam-strip parallel to the Ox_1 axis. If the deflection u is sought in the form $u = u(t, x_1)$, then in conformity with (2.11) we obtain an equation underlying the investigation in /3/

$$\beta\,[(I - R)\,u_{,11}]_{,11} + pu_{,11} = q$$

The cylindrical stiffness β here agrees with the bending stiffness of a rod of rectangular cross section of unit width.

6. Stability in a finite time interval. By analogy with /1–3/, we call the plate stable in an interval $[0, T]$ if $|u(t, x)| \leqslant \bar{u}$, $0 \leqslant t \leqslant T$, where $\bar{u}$ is the given critical value of the deflection. The time T_0 when the deflection first reaches the magnitude $\bar{u}$ is called critical.

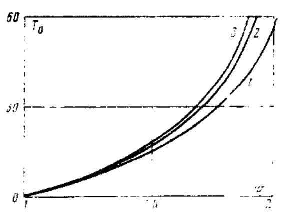
Let $p(x) \equiv 0$, and let ρ_0 denote the difference between the times of plate fabrication and load application (i.e., ρ_0 is the age of the plate material at the time of load application)

We investigage the influence of ρ_0 on the magnitude of the critical time T_0 by solving (2.11) numerically for a rectangular plate of length $a = 2$ m, width $b = 6$ m, and thickness $h = 0.2$ m, fabricated from alumina portland cement with the following parameters /14/:

$$k(t, \tau) = -E\frac{\partial}{\partial\tau}[\varphi(\tau)(1 - \exp(-\gamma(t - \tau)))],\quad \varphi(\tau) = A_0 + A_1/\tau,$$

$$E = 2.0\cdot 10^4\ \text{MPa},\ A_0 = 0.238\cdot 10^{-4}\ \text{MPa}^{-1},\ A_1 = 1.85\cdot 10^{-4}\ \text{MPa}^{-1}\ \text{day},$$

$$\nu = 0.333,\ \gamma = 0.04\ \text{day}^{-1}.$$



Constant compressive forces of strength $p = 7.5\cdot 10^4$ Pa are applied to the endfaces $|x_1| = a/2$. The endfaces $|x_2| = b/2$ are force-free. A transverse distributed load $q = q_0 \cos \pi x_2/b$, $q_0 = 833.33$ Pa is applied to the upper face of the plate. The dependence of the critical time T_0 on the maximal achievable value of the deflection $\bar{u}$ for different values of ρ_0 is represented in the figure where the quantity w, equal to the ratio between $\bar{u}$ and the maximum value of the elastic deflection at the time of external load application, is plotted along the abscissa axis, and the quantity T_0, measured in days, is plotted along the ordinate axis. Curve *1* corresponds to $\rho_0 = 10$ days, curve *2* to $\rho_0 = 20$ days, and curve *3* to $\rho_0 = 40$ days.

The results of the computations show that as ρ_0 grows the critical time T_0 increases, where the dependence of T_0 on the age of magnified as $\bar{u}$ increases.

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